

AN EXISTENCE RESULT

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ABSTRACT. On compact Riemannian manifold of dimension n , and under some conditions on the curvature, we have a nodal solutions for n large enough.

Introduction

Let (M, g) be a compact Riemannian manifold without boundary of dimension $n \geq 3$. We consider the following equation:

$$\Delta u = |u|^{2/(n-2)}u, \quad u \not\equiv 0 \quad (E)$$

Where $\Delta = -\nabla^i(\nabla_i)$ is the Laplace-Beltrami operator on M .

Holcman's Problem: Is there a nodal solution to the problem (E) ?

In his paper, see [3], Holcman proved that, if the scalar curvature R of M is positive somewhere, $(R(P) > 0, P \in M)$, then (E) has a nodal solution.

Here, we have,

Theorem. Assume that (M, g) is not conformally flat manifold of dimension $n \geq 13$ and $R \equiv 0$, then, (E) has a nodal solution.

For the proof of the Theorem, we use T. Aubin's and Holcman's methods and ideas and their computations. We use the variational method with an explicit expansion of the Yamabe type functional.

Question: Is it possible, if we use Schoen inequality, in the proof of the Yamabe problem for the conformally flat case, to have the same result for $R \equiv 0$?

Remark: Look also, the paper of E. Humbert and B. Ammann, [4], about "The second Yamabe invariant".

Proof of the Theorem.

Let us consider (M, g) a compact Riemannian manifold without boundary and not locally conformally flat. Assume that the scalar curvature $R \equiv 0$ and we work with the Yamabe functional:

$$J(\varphi) = \frac{\int_M |\nabla(\varphi)|^2 dV_g + \int_M R\varphi^2 dV_g}{\left(\int_M \varphi^N\right)^{2/N}} = \frac{\int_M |\nabla(\varphi)|^2 dV_g}{\left(\int_M \varphi^N\right)^{2/N}}.$$

Let P the point where $Weyl_g(P) \neq 0$,

As in the paper of T. Aubin, we do a conformal change of metric $\tilde{g} = \psi^{4/(n-2)}g$ such that:

$$\tilde{J}(\varphi_\epsilon) = \frac{1}{K} [1 - |Weyl_{\tilde{g}}(P)|^2 \epsilon^4 + o(\epsilon^4)],$$

where \tilde{J} is the Yamabe functional for the metric \tilde{g} and φ_ϵ the following functions:

$$\varphi_\epsilon(\tilde{r}) = \frac{\epsilon^{(n-2)/2}}{(\epsilon^2 + \tilde{r}^2)^{(n-2)/2}} - \frac{\epsilon^{(n-2)/2}}{(\epsilon^2 + \tilde{\delta}^2)^{(n-2)/2}} \quad \text{if } \tilde{r} = \tilde{d}(P, x) \leq \tilde{\delta}, \quad \text{otherwise } 0.$$

Also, we know that:

$$J(\psi\varphi_\epsilon) = \tilde{J}(\varphi_\epsilon).$$

Let us consider the following functions:

$$\bar{\varphi}_\epsilon = \psi(\varphi_\epsilon - \mu_\epsilon),$$

with, $\mu_\epsilon > 0$ is such that:

$$\int_M |\psi(\varphi_\epsilon - \mu_\epsilon)|^{N-2} [\psi(\varphi_\epsilon - \mu_\epsilon)] dV_g = 0.$$

If we compute with \tilde{g} , we have:

$$\int_M \frac{1}{\psi} |\varphi_\epsilon - \mu_\epsilon|^{N-2} (\varphi_\epsilon - \mu_\epsilon) d\tilde{V} = \int_M f |\varphi_\epsilon - \mu_\epsilon|^{N-2} (\varphi_\epsilon - \mu_\epsilon) d\tilde{V} = 0.$$

with $f = \frac{1}{\psi} > 0$.

We know, see Holcman, that μ_ϵ is equivalent to $\epsilon^{[(n-2)^2]/2(n+2)}$ for ϵ near 0.

Since the distance function \tilde{r} is Lipschitzian and equivalent to the first distance function r , we can compute (when we have the gradient), with respect to the \tilde{r} . We can write,

$$\int_M |\nabla[\psi(\varphi_\epsilon - \mu_\epsilon)]|^2 \leq \int_M |\nabla(\psi\varphi_\epsilon)|^2 dV_g + c_1\mu_\epsilon,$$

to see this, we write:

$$\begin{aligned} \int_M |\nabla[\psi(\varphi_\epsilon - \mu_\epsilon)]|^2 &= \int_M |\nabla(\psi\varphi_\epsilon)|^2 dV_g + 2\mu_\epsilon \int_M \langle \nabla\psi, \nabla(\psi\varphi_\epsilon) \rangle + O(\mu_\epsilon^2), \\ \int_M \langle \nabla\psi, \nabla(\psi\varphi_\epsilon) \rangle &= O\left(\int_M \varphi_\epsilon\right) + O\left(\int_M |\nabla(\varphi_\epsilon)|\right), \end{aligned}$$

We can see that;

$$\tilde{\nabla}^i(\varphi_\epsilon) = \psi^{-4/(n-2)} \nabla^i(\varphi_\epsilon),$$

Thus, for two positive constants C_1, C_2 , we have:

$$C_2 |\tilde{\nabla}\varphi_\epsilon| \leq |\nabla\varphi_\epsilon| \leq C_1 |\tilde{\nabla}\varphi_\epsilon| = C_1 |\partial_{\tilde{r}}\varphi_\epsilon(\tilde{r})|,$$

$$\int_M |\nabla\varphi_\epsilon| dV_g \leq C_4 \int_M |\partial_{\tilde{r}}\varphi_\epsilon(\tilde{r})| d\tilde{V} \leq C_5,$$

and,

$$\int_M |\nabla(\psi\varphi_\epsilon)|^2 dV_g = \int_M |\tilde{\nabla}\varphi_\epsilon|^2 d\tilde{V} + o(1) \geq C_6 \int_M |\partial_{\tilde{r}}\varphi_\epsilon(\tilde{r})|^2 d\tilde{V} \geq C_7 > 0,$$

(see, Aubin computations), and we have the result for the gradient.

And, we have:

$$\left(\int_M |\psi(\varphi_\epsilon - \mu_\epsilon)|^N\right)^{2/N} \geq \left(\int_M |\psi\varphi_\epsilon|^N\right)^{2/N} - c_2\mu_\epsilon, \text{ with } c_2 > 0.$$

because,

$$\|\psi(\varphi_\epsilon - \mu_\epsilon)\|_{L^N, g}^N = \int_M |\psi(\varphi_\epsilon - \mu_\epsilon)|^N dV = \int_M |(\varphi_\epsilon - \mu_\epsilon)|^N d\tilde{V} = \|(\varphi_\epsilon - \mu_\epsilon)\|_{L^N, \tilde{g}}^N,$$

and, for \tilde{g}

$$\|\varphi_\epsilon\|_{L^N, \tilde{g}} \leq \|(\varphi_\epsilon - \mu_\epsilon)\|_{L^N, \tilde{g}} + |M|^{1/N} \mu_\epsilon,$$

and, because $\|(\varphi_\epsilon - \mu_\epsilon)\|_{L^N, \tilde{g}} \rightarrow c > 0$ (or, $\|\varphi_\epsilon\|_{L^N, \tilde{g}} \rightarrow c' > 0$.) (see the computations of Holcman's paper with the metric \tilde{g}),

$$\|\psi\varphi_\epsilon\|_{L^N, g}^2 = \|\varphi_\epsilon\|_{L^N, \tilde{g}}^2 \leq \|(\varphi_\epsilon - \mu_\epsilon)\|_{L^N, \tilde{g}}^2 + c_2\mu_\epsilon,$$

and then,

$$\frac{\int_M |\nabla[\psi(\varphi_\epsilon - \mu_\epsilon)]|^2}{(\int_M |\psi(\varphi_\epsilon - \mu_\epsilon)|^N)^{2/N}} \leq J(\psi\varphi_\epsilon)(1 + c_3\mu_\epsilon)$$

Thus,

$$\frac{\int_M |\nabla[\psi(\varphi_\epsilon - \mu_\epsilon)]|^2}{(\int_M |\psi(\varphi_\epsilon - \mu_\epsilon)|^N)^{2/N}} \leq \frac{1}{K}[1 + c_4\epsilon^{(n-2)^2/2(n+2)} - |Weyl_{\tilde{g}}(P)|^2\epsilon^4 + o(\epsilon^4)].$$

We can say that, $\epsilon^{(n-2)^2/2(n+2)}$ is very small if we compare it to ϵ^4 if, $\frac{(n-2)^2}{2(n+2)} > 4$, and then, if $n \geq 13$.

Thus, on M , we have test functions

$$\bar{\varphi}_\epsilon = \psi(\varphi_\epsilon - \mu_\epsilon) \neq 0,$$

such that:

$$\int_M |\bar{\varphi}_\epsilon|^{N-2} \bar{\varphi}_\epsilon dV_g = 0,$$

and, the Sobolev quotient is such that:

$$\frac{\int_M |\nabla \bar{\varphi}_\epsilon|^2}{(\int_M |\bar{\varphi}_\epsilon|^N)^{2/N}} \leq \frac{1}{K}[1 + c_4\epsilon^{(n-2)^2/2(n+2)} - |Weyl_{\tilde{g}}(P)|^2\epsilon^4 + o(\epsilon^4)] < \frac{1}{K}.$$

Thus, the variational problem has a nodal solution on M .

Remark 1: We can replace μ_ϵ by μ_ϵ^2 , in this case we can assume $n \geq 9$.

Remark 2: This method works if we assume that, there is a point P such that $Weyl_g(P) \neq 0$ and $R \equiv 0$ in the neighborhood of P . (Such manifolds exist, it is sufficient to solve the prescribed scalar curvature problem for non-positive scalar curvature, by considering the condition on the first eigenvalue of small balls, see Rauzy and Veron in [1]).

Remark 3: This method works if we assume that, there is a point P such that $Weyl_g(P) \neq 0$ and $R(P) = \nabla R(P) = \nabla^2 R(P) = 0$.

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